# Sharp Bounds for Exponential Approximations of NWUE Distributions

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# Outline

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1. Introduction
   - Backgrounds
   - The Problem of Interest
2. Main Results
   - Sharp Bounds on $K(W^*, (EW)\mathcal{E})$
   - Sharp Bounds on $K(W^*, (EW^*)\mathcal{E})$
3. Applications and Discussions
   - Applications
   - Discussions

# Exponential Approximations

Often in applied probability models an intractable random variable of interest is plausibly approximately exponentially distributed. This can often be argued by **limit theorems**, but more convincingly by small **error bounds** on the distance (Kolmogorov or total variation), to an approximating exponential distribution.

# Aging Distributions

Frequently in applications the distribution of interest is known to belong to a class of aging distributions. The mathematical challenge is to obtain sharp error bounds for that class given the first two moments of the distribution.

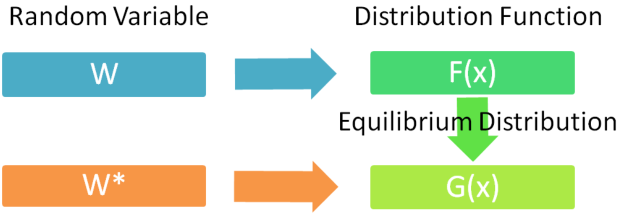
- Assume $W \geq 0$ is a random variable with $EW^2 < \infty$. The distribution of $W$ is said to be **NWUE** (new worse than used in expectation) if $E(W - t|W > t) \geq EW$ for all $t \geq 0$.
- Assume $W \geq 0$ is a random variable with $EW^2 < \infty$. The distribution of $W$ is said to be **NBUE** (new better than used in expectation) if $E(W - t|W > t) \leq EW$ for all $t \geq 0$.

# Equilibrium Distribution

Let $W$ be a non-negative random variable with finite first and second moments. Let $F$ be the distribution function of $W$. $G$, the equilibrium distribution of $F$, is defined as

$$\bar{G}(x) = \frac{1}{EW} \int_x^\infty \bar{F}(t)\, dt,$$

where $\bar{G}(x) = 1 - G(x)$ and $\bar{F}(x) = 1 - F(x)$.
Let $W^*$ be a random variable with distribution function $G$.



# Equilibrium Distribution

- The distribution of $W$ being NWUE is equivalent to $\bar{G}(x) \geq \bar{F}(x)$. The distribution of $W$ being NBUE is equivalent to $\bar{G}(x) \leq \bar{F}(x)$.
  *Proof(NWUE).*
  $$\bar{G}(x) = \frac{1}{EW} \int_x^\infty \bar{F}(t)\ dt$$
  $$\begin{aligned} \Rightarrow \bar{G}(x)/\bar{F}(x) &= \frac{1}{EW}\Big( \int_x^\infty \bar{F}(t)\ dt\ / \bar{F}(x) \Big) \\ &= \frac{1}{EW} E(W - x | W > x) \geq 1. \end{aligned}$$
- $W \sim$ Exponential $\Rightarrow W^* \sim$ Exponential.
- $W$ and $W^*$ have the same distribution $\Rightarrow W \sim$ Exponential.

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# The Problem of Interest

- Let $W$ be a random variable belonging to the NWUE family. Let $\mathcal{E}$ be a Exponential(1) random variable.
- $K(W^*, (EW)\mathcal{E})$ – Kolmogorov distance between $W^*$ and an exponential distribution with the same mean as $W$
  $K(W^*, (EW^*)\mathcal{E})$ – Kolmogorov distance between $W^*$ and an exponential distribution with the same mean as $W^*$
- The goal is to obtain sharp bounds given the first two moments of the distribution. For NBUE or NWUE and their subclasses, Keilson(1979) suggested the scale invariant parameter $|\rho|$, where

$$\rho = \frac{EW^2}{2(EW)^2} - 1 = \frac{\mu_G}{\mu_F} - 1.$$

Hence the bounds obtained should be written as functions of $\rho$.

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# Sharp Bounds on $K(W^*, (EW)\mathcal{E})$

**Theorem**

$$K(W^*, (EW)\mathcal{E}) \leq p + q \log q,$$

*where*

$$p = \sqrt{\rho^2 + 2\rho} - \rho,$$

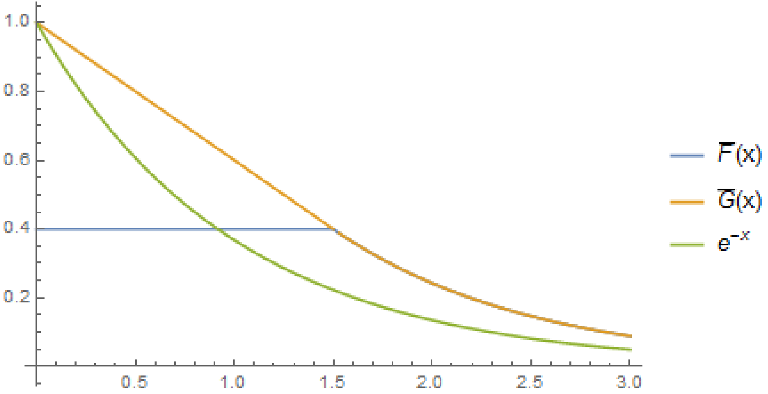
*and* $q = 1 - p$. *The bound is sharp.*

- Note that as $W^* \overset{st}{\geq} (EW)\mathcal{E}$, one-sided Kolmogorov distance is not an issue.

# Example – Attaining the Bound

## Example

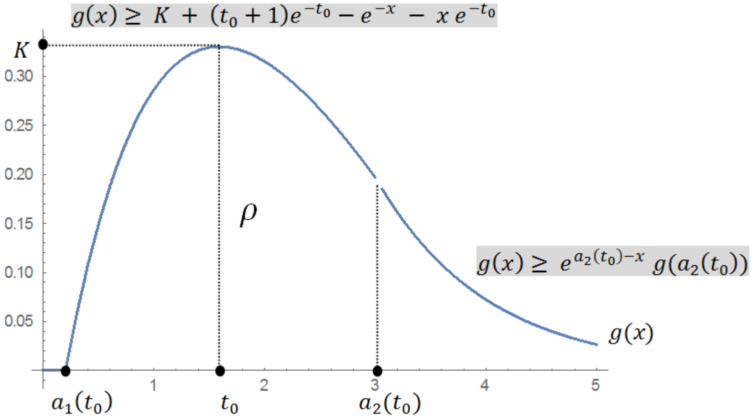
- $$\bar{F}(x) = \begin{cases} 1 - p & \text{if } x < \frac{p}{1-p}, \\ (1-p)\ e^{p/(1-p)-x} & \text{if } x \geq \frac{p}{1-p}. \end{cases}$$



# Idea Behind the Proof

Let $g(x) = \bar{G}(x) - e^{-x}$. $W$ follows an NWUE distribution implies that $g(x) \geq 0$, and that,

$$\int_0^\infty g(x)dx = \rho.$$



# Outline

# Sharp Bounds on $K^+(W^*, (EW^*)\mathcal{E})$

**Theorem**

$$K^+(W^*, (EW^*)\mathcal{E}) = \sup_t \left[\bar{G}(t) - e^{-t/EW^*}\right] \leq p^* + q^* \log q^*,$$

*where $q^* = q(1+\rho)$ and $p^* = 1 - q^*$. The bound is sharp.*

- If $W$ is not exponentially distributed, then $p^* < p$ and the bound is smaller than $p + q \log q$, i.e. the sharp bound for $K(W^*, (EW)\mathcal{E})$.
- For example if $\rho = 0.1$, then $p + q \log q = 0.0736$, while $p^* + q^* \log q^* = 0.0482$.
- The example in the previous section also attains the bound here. $\Rightarrow$ The bound is sharp.

# Sharp Bounds on $K^-(W^*, (EW^*)\mathcal{E})$

**Theorem**

$$K^-(W^*, (EW^*)\mathcal{E}) = \sup_t \left[e^{-t/EW^*} - \bar{G}(t)\right] \le \frac{\rho}{1+\rho}\Big(\frac{1}{\rho+1}\Big)^{1/\rho}.$$

*The bound is sharp.*

**Proof.**

For $K^-$, since $\bar{G}(x) \ge e^{-x}$,

$$K^- = \sup_x \{e^{-x/(1+\rho)} - \bar{G}(x)\} \le \sup_x \{e^{-x/(1+\rho)} - e^{-x}\} = \Big(\frac{\rho}{1+\rho}\Big)\Big(\frac{1}{\rho+1}\Big)^{\frac{1}{\rho}}.$$

□

# Example – Attaining the Bounds on $K^-(W^*, (EW^*)\mathcal{E})$

## Example

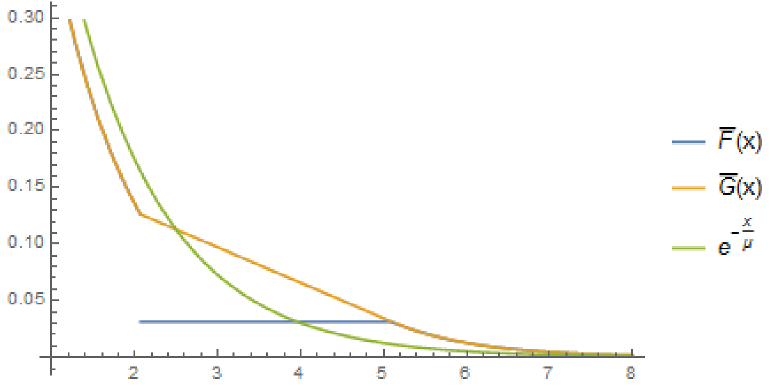



Figure: Example – Attaining the Bound

# Combine the Results – Sharp Bounds on $K(W^*, (EW^*)\mathcal{E})$

**Theorem**

$$K(W^*, (EW^*)\mathcal{E}) \leq \max\left(p^* + q^* \log q^*, \frac{\rho}{\rho+1}\left(\frac{1}{\rho+1}\right)^{1/\rho}\right)$$
$$= \begin{cases} p^* + q^* \log q^* & \text{if } \rho \leq c, \\ \frac{\rho}{\rho+1}\left(\frac{1}{\rho+1}\right)^{1/\rho} & \text{if } \rho > c, \end{cases}$$

*where* $c \approx 0.245018$*. The bound is sharp.*

- For any $\rho > 0$, the bound is strictly smaller than the bound for $K(W^*, (EW)\mathcal{E})$.

# Outline

# Applications

- Geometric Convolutions.
  Let $\{X_i\}_{i\geq 0}$ be an i.i.d sequence of non-negative random variables with finite mean $\nu_1$ and finite second moment $\nu_2$. Let $\theta$ be in $(0,1)$. Let $N$ follow a geometric distribution and be independent of the sequence with $P(N=k)=(1-\theta)^k\theta,\ k\in\mathbb{Z}, k\geq 0$.

$$Y=\sum_{i=1}^{N} X_i$$

  is called a geometric convolution.
  Brown (1990) showed that $Y$ is NWU, and as $Y$ has a finite mean, $Y$ is NWUE.
- First Passage Times.

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# Comparison of Bounds – Bounded Hazard Rate

Brown (2015) considered sharp bounds for exponential approximations under a hazard rate upper bound. As $W$ here has an NWUE distribution, the hazard rate of $W^*$ is bounded above by 1. Hence by Brown's results, $K(W^*, \mathcal{E}) \leq 1 - e^{-\rho}$, and $K(W^*, (1+\rho)\mathcal{E}) \leq 1 - e^{-\frac{\rho}{1+\rho}}$ for $\rho$ small.

| | Bound for $K(W^*, \mathcal{E})$ | | Bound for $K(W^*, (1+\rho)\mathcal{E})$ | |
|---|---|---|---|---|
| $\rho$ | $q \log q + p$ | $1 - e^{-\rho}$ | $1 - \mu q + \mu q \log(\mu q)$ | $1 - e^{-\frac{\rho}{1+\rho}}$ |
| 0.005 | 0.0047 | 0.0050 | 0.0042 | 0.0050 |
| 0.01 | 0.0091 | 0.0100 | 0.0079 | 0.0099 |
| 0.02 | 0.0175 | 0.0198 | 0.0144 | 0.0194 |
| 0.05 | 0.0403 | 0.0488 | 0.0297 | 0.0465 |
| 0.1 | 0.0738 | 0.0952 | 0.0482 | 0.0869 |
| 0.2 | 0.1293 | 0.1813 | 0.0726 | 0.1535 |

Table: Comparison of Bounds

# Lower Bounds

- There are no positive lower bounds for either $K(W^*, E(W)\mathcal{E})$ or $K(W^*, E(W^*)\mathcal{E})$.
- For $K(W^*, E(W^*)\mathcal{E})$, if $\bar{F}(t) = \frac{1}{\rho+1}e^{-t/(\rho+1)}$, then $W^* \sim (1+\rho)\mathcal{E}$ and $K(W^*, (1+\rho)\mathcal{E}) = 0$.
- For $K(W^*, E(W)\mathcal{E})$, for any $\rho$ and $\epsilon > 0$, define,

$$\bar{F}(t) = \begin{cases} e^{-t}, \ 0 \leq t < T, \\ e^{-T}(qe^{-q(t-T)}) = qe^{-pT}e^{-qt}, \ t \geq T, \end{cases}$$

where $q$ is chosen to be $\frac{1}{1+\rho e^T}$ and T to be $\log(\frac{1}{\epsilon})$.
$\Rightarrow K(W^*, E(W)\mathcal{E}) < \epsilon$.

# Summary

- Limit theorems and error bounds are frequently proved for exponential approximations. Aging distributions are often of interest. Equilibrium distributions are closely related to exponential approximations and aging distributions.
- Sharp upper bounds are obtained on $K(W^*, E(W)\mathcal{E})$ and $K(W^*, E(W^*)\mathcal{E})$
- The results (sharp bounds) can be applied to geometric convolutions and first passage times.
- There are no positive lower bounds for either $K(W^*, E(W)\mathcal{E})$ or $K(W^*, E(W^*)\mathcal{E})$.

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# Thank you!